KUWAIT UNIVERSITY
Department of Mathematics & Computer Science

Math 102
Calculus II

Final Exam

23 January 2006
Time: 2 hours

Calculators and cellular phones are not allowed during the exam.

1. Let $f(x) = \tan^{-1}(x + 2\ln x)$, where $x > 0$. (a) Show that f is one-to-one on its domain. (b) Find the range of f. (c) Find the equation of the tangent line to the graph of f^{-1} at the point $P(\frac{\pi}{4}, 1)$. [6 pts.]

2. Prove the identity: [3 pts.]

$$\log_x(xy)\,\log_y(xy) = \log_x y + \log_y x + 2, \quad \text{for any positive } x, y \neq 1$$

3. Find the limit: $\lim\limits_{x\to 0} \dfrac{x\tan^{-1}x}{\sin^{-1}x}$ [3 pts.]

4. Evaluate the following integrals: [9 pts.]

$$\text{(a)} \int \frac{x}{\sqrt{x^2 - 4x + 3}}\,dx \qquad \text{(b)} \int_0^{\pi/8} (\cos 3x)(\cos 5x)\,dx \qquad \text{(c)} \int e^x \cos 2x\,dx$$

5. Determine whether the following integral is convergent or divergent and find its value i case of convergence: [4 pts.

$$\int_{\ln 3}^{\infty} \frac{e^x}{e^{2x} - 3e^x + 2}\,dx$$

6. Let C be a curve defined by the equation $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, x \in [1, 2]$. (a) Find the length of C. (b) Find the area of the surface obtained by rotating C about the y-axis. [3 + 2 pts.

7. Let Γ be a plane curve with parametrization: $x = t - e^t$, $y = t + e^t$, where $t \in \mathbb{R}$. [6 pts

   a. Write the equation of the tangent line to Γ at the point corresponding to $t = 0$.

   b. Find the points on Γ (if any) where the tangent lines are vertical and those where th tangent lines are horizontal.

   c. Find the intervals on which this curve is concave upward and those where it is conca downward.

8. Find the area of the region that lies inside both curves $r = \frac{1}{2}$ and $r = \cos 2\theta$. [4 pt

Q1 $f(x) = \tan^{-1}(x + 2\ln x)$, $D_f = (0, \infty)$ given $R_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b)

$f'(x) = \frac{1}{1+(x+2\ln x)^2}\left(1+\frac{2}{x}\right) > 0$ in D_f ↑ ∴ 1-1 function

c) $(f^{-1})'(\pi/4) = \frac{1}{f'[f^{-1}(\frac{\pi}{4})]} = \frac{1}{f'(1)}$ As $f'(1) = \frac{1}{2} \cdot 3 = \frac{3}{2}$.

∴ $(f^{-1})'(\frac{\pi}{4}) = 2/3$ ∴ Eq of tang. line $y - 1 = \frac{2}{3}(x - \frac{\pi}{4})$ An

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Q2 L.H.S $\log_x(xy)\log_y(xy) = \left(\log_x x + \log_x y\right)\left(\log_y x + \log_y y\right)$

$= (1 + \log_x y)(\log_y x + 1) = \log_y x + 1 + \log_x y \cdot \log_y x + \log_x y$.

As $\log_x y \cdot \log_y x = 1$

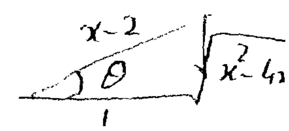
L.H.S $= \log_y x + 1 + 1 + \log_x y = \log_y x + \log_x y + 2$ R.H.S

Q3 $\lim_{x \to 0} \frac{x\tan^{-1}x}{\sin^{-1}x}$ $\frac{0}{0}$, $\lim_{x\to 0} \frac{\tan^{-1}x + \frac{x}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{0}{1} = 0$

Q4 (a) $\int \frac{x}{\sqrt{x^2-4x+3}}dx = \int \frac{x}{\sqrt{(x-2)^2-1}}dx$ let $x - 2 = \sec\theta$, $dx = \sec\theta\tan\theta d\theta$

$= \int \frac{2+\sec\theta}{\cancel{\tan\theta}} \cdot \sec\theta \cancel{\tan\theta} d\theta = 2\int \sec\theta d\theta + \int \sec^2\theta d\theta$

$= 2\ln|\sec\theta + \tan\theta| + \tan\theta + C$



$= 2\ln|x - 2 + \sqrt{x^2-4x+3}| + \sqrt{x^2-4x+3} + C$ Ans

b) $\frac{1}{2}\int_0^{\pi/8} 2\cos 5x\cos 3x\, dx = \frac{1}{2}\int_0^{\pi/8}(\cos 8x + \cos 2x)dx = \frac{1}{2}\left|\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left[\frac{1}{2\sqrt{2}} - 0\right] = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$ Ans

c) $I = \int e^x\cos 2x\,dx = \frac{e^x\sin 2x}{2} - \frac{1}{2}\int e^x\sin 2x\,dx = \frac{e^x\sin 2x}{2} - \frac{1}{2}\left[-\frac{e^x\cos 2x}{2} + \frac{1}{2}\right.$

$I = \frac{e^x\sin 2x}{2} + \frac{1}{4}e^x\cos 2x - \frac{1}{4}I$

$\Rightarrow \frac{5}{4}I = \frac{e^x\sin 2x}{2} + \frac{e^x\cos 2x}{2} \Rightarrow I = \frac{4}{5}\left[\frac{e^x\sin 2x}{2} + \frac{e^x\cos 2x}{4}\right] +$

Q5 $\int_{\ln 3}^{\infty} \frac{e^x}{e^{2x} - 3e^x + 2}dx$ let $e^x = u$ $e^x dx = du$ $= \int_3^{\infty}\frac{1}{(u-1)(u-2)}du$

$= \lim_{t\to\infty}\int_3^t \frac{1}{(u-2)(u-1)}du = \lim_{t\to\infty}\int_3^t\left(\frac{1}{u-2} - \frac{1}{u-1}\right)du = \lim_{t\to\infty}\left[\ln\left|\frac{u-2}{u-1}\right|\right]_3^t$

$= \lim_{t\to\infty}\left[\ln\left|\frac{t-2}{t-1}\right| - \ln\frac{1}{2}\right] = \ln 2$ convergent

Q6. $y = \frac{x^3}{3} + \frac{1}{4x}$, $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$, $\left(\frac{dy}{dx}\right)^2 = x^4 + \frac{1}{16x^4} - \frac{1}{2}$ ∴

$1 + \left(\frac{dy}{dx}\right)^2 = x^4 + \frac{1}{16x^4} + \frac{1}{2} = \left(x^2 + \frac{1}{4x^2}\right)^2$.

$L = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right)dx = \left|\frac{x^3}{3} - \frac{1}{4x}\right|_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{7}{3} - \frac{1}{8} + \frac{1}{4} = \frac{7}{3} + \frac{-2+1}{8}$ ... $= \frac{7}{3} + \frac{1}{8} = \frac{56+3}{24} = \frac{59}{24}$ Ans

b) $S = \int_1^2 2\pi \cdot x \left(x^2 + \frac{1}{4x^2}\right)dx = 2\pi \int_1^2 \left(x^3 + \frac{1}{4x}\right)dx = 2\pi \left|\frac{x^4}{4} + \frac{1}{4}\ln x\right|_1^2$

$S = 2\pi\left[4 + \frac{1}{4}\ln 2 - \frac{1}{4}\right] = 2\pi\left(\frac{15}{4} + \frac{1}{4}\ln 2\right) = \frac{\pi}{2}(15 + \ln 2)$

Ans

Q7 a) $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{1+e^t}{1-e^t}$, m at $t=0 = \infty$.

$x = -1$, $y = 1$ at $t = 0$ ∴ Eq of tangent line

$\frac{y - y_1}{m} = x - x_1 \Rightarrow \frac{y - y_1}{\infty} = x - x_1 \Rightarrow x - x_1 = 0 \Rightarrow x + 1 = 0$

b) Vertical $\frac{dx}{dt} = 0$ $\quad 1 - e^t = 0 \Rightarrow e^t = 1 \quad t = 0$.

Pt is $(-1, 1)$.

Horizontal $\frac{dy}{dt} = 0 \quad 1 + e^t = 0 \Rightarrow e^t = -1$ not possible

Non pt for horizontal tangent line

c) $\frac{dy}{dx} = \frac{1+e^t}{1-e^t}$ ∴ $\frac{d^2y}{dx^2} = \frac{(1-e^t)e^t\frac{dt}{dx} + (1+e^t)e^t\frac{dt}{dx}}{(1-e^t)^2}$

$= \frac{e^t\frac{dt}{dx}\left[1 - \cancel{e^t} + 1 + \cancel{e^t}\right]}{(1-e^t)^2} = \frac{2e^t}{(1-e^t)^2} \cdot \frac{1}{1-e^t} = \frac{2e^t}{(1-e^t)^3}$

$t=0$

	$t<0$	$t>0$
$2e^t$	$++++$	$++++$
$(1-e^t)^3$	$++++$	$----$
	$+$	$-$

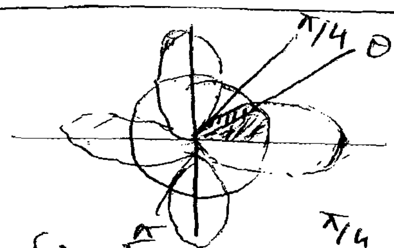
Concave up $(-\infty, 0)$

Concave down $(0, \infty)$ t

Q8 $r = \frac{1}{2}$ $\quad r = \cos 2\theta$

$\frac{1}{2} = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$



$A = \frac{8}{2}\left[\int_0^{\pi/6} \frac{1}{4}\,d\theta + \int_{\pi/6}^{\pi/4} \cos^2 2\theta\,d\theta\right] = 4\left[\frac{1}{4}\left[\theta\right]_0^{\pi/6} + \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/6}^{\pi/4}\right]$

$= 4\left[\frac{1}{4}\left(\frac{\pi}{6}\right) + \frac{1}{2}\left[\frac{\pi}{4} - \frac{\pi}{6} - \frac{1}{4}\frac{\sqrt{3}}{2}\right]\right]$

$= 4\left[\frac{\pi}{24} + \frac{\pi}{8} - \frac{\pi}{12} - \frac{1}{16}\sqrt{3}\right] = \frac{\pi}{6} + \frac{\pi}{2} - \frac{\pi}{3} - \frac{1}{4}\sqrt{3} = \frac{\pi + 3\pi - 2\pi}{6} - \frac{\sqrt{3}}{4}$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$ Ans

SOLUTIONS

1. (a) $f'(x) = \frac{1}{1+(x+2\ln x)^2}\left(1+\frac{2}{x}\right) > 0$, for $x>0 \Rightarrow f$ is increasing $\Rightarrow f$ is 1-1 on $(0,\infty$

(b) $\lim_{x\to 0^+} \tan^{-1}(x+2\ln x) = -\frac{\pi}{2}$, $\lim_{x\to\infty}\tan^{-1}(x+2\ln x)=\frac{\pi}{2}$, so $R_f = \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $m = (f^{-1})'\left(\frac{\pi}{4}\right) = \frac{1}{f'(1)} = \frac{2}{3}$, so eq. of tangent line: $y-1=\frac{2}{3}\left(x-\frac{\pi}{4}\right)$. (2

2. $\log_x(xy)\log_y(xy) = (1+\log_x y)(1+\log_y x) = 1+\log_x y+\log_y x+\log_x y\log_y x$

$= 2+\log_x y+\log_y x$ (because: $\log_x y \log_y x = \frac{\ln y}{\ln x}\frac{\ln x}{\ln y} = 1$). (3P

3. $\lim_{x\to0}\frac{x\tan^{-1}x}{\sin^{-1}x} = \frac{0}{0}$ (I.F.), Apply L'H Rule: $\lim_{x\to0}\frac{x\tan^{-1}x}{\sin^{-1}x} = \lim \frac{\tan^{-1}x+\frac{x}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}}$ (3

4. (a) $I=\int\frac{x}{\sqrt{(x-2)^2-1}}dx = \int\frac{2+u}{\sqrt{u^2-1}}du = 2\int\frac{du}{\sqrt{u^2-1}}+\int\frac{u\,du}{\sqrt{u^2-1}}$
$(x-2=u, dx=du)$.

$=2\ln(u+\sqrt{u^2-1})+\sqrt{u^2-1}+C = 2\ln(x-2+\sqrt{x^2-4x+3})+\sqrt{x^2-4x+3}+C$

(b) $\int_0^{\pi/8}\cos3x\cos5x\,dx = \frac{1}{2}\int_0^{\pi/8}(\cos8x+\cos2x)dx = \frac{1}{16}\sin8x+\frac{1}{4}\sin2x\Big|_0^{\pi/8} = \frac{\sqrt{2}}{8}$. (3P)

(c) $I = \frac{1}{2}e^x\sin2x - \frac{1}{2}\int e^x\sin2x\,dx = \frac{1}{2}e^x\sin2x - \frac{1}{2}\left(-\frac{1}{2}e^x\cos2x+\frac{1}{2}\int e^x\cos2x\,dx\right.$
$\left(u=e^x, dv=\cos2x\,dx, du=e^x dx, v=\frac{1}{2}\sin2x\right)$ $\left(u=e^x, dv=\sin2x\,dx, du=e^x dx, v=-\frac{1}{2}\cos2x\right)$

$=\frac{1}{2}e^x\left(\sin2x+\frac{1}{2}\cos2x\right)-\frac{1}{4}I \Rightarrow I=\frac{2}{5}e^x\left(\sin2x+\frac{1}{2}\cos2x\right)+C$. (3P)

5. $\int\frac{e^x}{e^{2x}-3e^x+2}dx = \int\frac{du}{u^2-3u+2} = \int\left(\frac{1}{u-2}-\frac{1}{u-1}\right)du = \ln\left|\frac{u-2}{u-1}\right| = \ln\left|\frac{e^x-2}{e^x-1}\right|$.
$(e^x=u, e^x dx=du)$

$\lim_{t\to\infty}\int_{\ln3}^t \frac{e^x}{e^{2x}-3e^x+2}dx = \lim_{t\to\infty}\left(\ln\frac{e^t-2}{e^t-1} - \ln\frac{e^{\ln3}-2}{e^{\ln3}-1}\right) = \ln1-\ln\frac{1}{2}=\ln2$. (

6. (a) $y'=x^2-\frac{1}{4x^2}$, $1+(y')^2 = 1+x^4-\frac{1}{2}+\frac{1}{16x^4} = x^4+\frac{1}{2}+\frac{1}{16x^4} = \left(x^2+\frac{1}{4x^2}\right.$

$\Rightarrow L=\int_1^2\sqrt{1+(y')^2}\,dx = \int_1^2\left(x^2+\frac{1}{4x^2}\right)dx = \frac{59}{24}$. (3P).

(b) $A=2\pi\int_1^2 x\sqrt{1+(y')^2}\,dx = 2\pi\int_1^2\left(x^3+\frac{1}{4x}\right)dx = 2\pi\left(\frac{x^4}{4}+\frac{1}{4}\ln x\right)\Big|_1^2 = \frac{\pi}{2}(15+\ln2)$. (2P

7. (a) $\frac{dx}{dt}=1-e^t$, $\frac{dy}{dt}=1+e^t \Rightarrow \frac{dx}{dt}\Big|_{t=0}=0$, $\frac{dy}{dt}\Big|_{t=0}=2$. The point is $P(-1,1)$. The tangent line is vertical and its equation is $x=-1$. (2P)

(b) Vertical when: $\frac{dx}{dt}=0$, $\frac{dy}{dt}\neq0 \Rightarrow 1-e^t=0\Rightarrow t=0\Rightarrow P(-1,1)$

(2P) Horizontal: $\frac{dy}{dt}=0$, $\frac{dx}{dt}\neq0\Rightarrow 1+e^t=0$ (no solution) $\Rightarrow$ no point

(c) $\frac{dy}{dx}=\frac{1+e^t}{1-e^t}$, $\frac{d^2y}{dx^2}=\frac{2e^t}{(1-e^t)^3}$ $\Rightarrow$ concave upward for $t\in(-\infty,0)$, concave downward for $t\in(0,\infty)$. (2P

8. $r=\frac{1}{2}$ and $r=\cos2\theta \Rightarrow \cos2\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$. $P\left(\frac{1}{2},\frac{\pi}{6}\right)$.

Area R = 8 (Area R_1 + Area R_2) $\left(\cos2\theta=0\Rightarrow2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}\right)$.

$=8\left(\frac{1}{2}\int_0^{\pi/6}\frac{1}{4}d\theta+\frac{1}{2}\int_{\pi/6}^{\pi/4}\cos^2 2\theta\,d\theta\right)$

$=\frac{\pi}{6}+2\int_{\frac{\pi}{6}}^{\pi/4}(1+\cos4\theta)\,d\theta$

$=\frac{\pi}{6}+2\left(\theta+\frac{1}{4}\sin4\theta\right)\Big|_{\frac{\pi}{6}}^{\pi/4}=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$. (4P)